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NOTE

Notes on Steklov's Conjecture in L^p and on Divergence of Lagrange Interpolation in L^p*

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Given a compact interval Δ, it is shown that for E. A. Rakhmanov's weight w on Δ which is bounded from below by the Chebyshev weight v on Δ (1982, *Math. USSR Sb.* **42**, 263) the corresponding orthonormal polynomials are unbounded in every L^p_v (and L^p_w) with $p>2$ and also that the Lagrange interpolation process based on their zeros diverges in every L^p_v with $p>2$ for some continuous f. This yields an affirmative answer to Conjecture 2.9 in "Research Problems in Orthogonal Polynomials" (1989, *in* "Approximation Theory, VI," Vol. 2, p. 454; (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), Academic Press, New York) a positive answer to Problem 8, and a negative answer to Problem 10 of P. Turán (1980, *J. Approx. Theory* **29**, 32–33).

Throughout this note, let Δ be a fixed compact interval of positive length, say $\Delta =^{\text{def}} [a, b]$ with $-\infty < a < b < \infty$. The function w is called a weight in Δ if $w(\geqslant 0) \in L^1(\Delta)$ and $\int_\Delta w > 0$. Given a weight w in Δ, let $\{p_n(w)\}_{n=0}^{\infty}$ $(\deg(p_n) = n)$ denote the system of orthonormal polynomials with respect to w. Given $n \in \mathbb{N}$ and $f \in \mathbf{C}(\Delta)$, the Lagrange interpolating

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polynomial $L_n(w, f)$ based on the zeros of p_n is defined as the unique polynomial of degree $n-1$ that takes the values of f at the zeros of $p_n(w)$.

In what follows, we will use the notation

$$\|g\|_p \overset{\text{def}}{=} \left\{\int_\Delta |g|^p\right\}^{1/p}, \qquad \|g\|_{u,p} \overset{\text{def}}{=} \left\{\int_\Delta |g|^p\, u\right\}^{1/p}, \qquad \|g\|_\infty \overset{\text{def}}{=} \max_\Delta |g|,$$

and

$$\|L_n(w)\|_{u,p} \overset{\text{def}}{=} \sup_{\|f\|_\infty = 1} \|L_n(w, f)\|_{u,p}$$

where $u \geqslant 0$ in Δ and $0 < p < \infty$. The Chebyshev weight corresponding to Δ is denoted by v, that is,

$$v(x) \overset{\text{def}}{=} \frac{1}{\pi\sqrt{(b-x)(x-a)}}, \qquad x \in \Delta = [a, b].$$

For mean convergence of Lagrange interpolation the first two significant results, one of them due to P. Erdős and P. Turán [2], and the other one due to Erdős and E. Feldheim [1], are as follows.[1]

THEOREM A. *Given a weight* w *in* Δ, $\lim_{n\to\infty} \|f - L_n(w, f)\|_{w,2} = 0$ *holds for all* $f \in \mathbf{C}(\Delta)$.

THEOREM B. *If* $0 < p < \infty$ *then* $\lim_{n\to\infty} \|f - L_n(v, f)\|_{v,p} = 0$ *holds for all* $f \in \mathbf{C}(\Delta)$.

In 1975, Turán restated two of his favorite problems in this direction in [9, p. 32–33].

PROBLEM 8. Are there a weight w in Δ and a function $f \in \mathbf{C}(\Delta)$ such that we have $\limsup_{n\to\infty} \|f - L_n(w, f)\|_{w,p} = \infty$ for every $p > 2$?

PROBLEM 10 (Erdős–Feldheim). Let $w \geqslant v$ in Δ. Is it true that if $p > 0$ and $f \in \mathbf{C}(\Delta)$ then $\lim_{n\to\infty} \|f - L_n(w, f)\|_p = 0$?

Problem 8 was first solved in [3, Corollary 10.18, p. 181] and [4, Theorem, p. 190], and then improved in [8, Corollary 14, p. 326] as follows.

[1] Most of the subsequent problems, conjectures, and results were originally formulated for the case when $\Delta = [-1, 1]$, but, of course, they are equivalent to the case when Δ is an arbitrary compact interval.

THEOREM C. *Let* $2 \leqslant p_0 < \infty$, *and let* w *and* u *be weights in* Δ. *If* $\int_\Delta [w/v]^{-p/2} u = \infty$ *holds for every* $p > p_0$, *then there exists a function* $f \in \mathbf{C}(\Delta)$ *such that* $\limsup_{n\to\infty} \|f - L_n(w, f)\|_{u,p} = \infty$ *holds for every* $p > p_0$.

Unfortunately, this theorem gives answers neither to Problem 8 with the additional condition $w \geqslant v$ in Δ nor to Problem 10, since if $w \geqslant v$ in Δ then $\int_\Delta (w/v)^{-p/2} u \leqslant \int_\Delta u < \infty$.

Closely related to these problems is a conjecture given in [5, Conjecture 2.9, p. 454] which is related to Steklov's conjecture (see, for instance, [9, Problems 68 and 69, p. 60] and [6, p. 549]). Since E. A. Rakhmanov disproved the original conjecture of Steklov in [6, Theorem 2, p. 566]), it is natural to expect that its L^p_v variant with all $p > 2$ fails as well. Hence we have the following.

CONJECTURE 2.9. Given $p > 2$, there exists a weight w in Δ such that $w \geqslant v$ in Δ and the sequence $\{p_n(w)\}_{n=0}^{\infty}$ is unbounded in L^p_w.

The purpose of this note is to give a positive answer to Problem 8 with the additional condition $w \geqslant v$ in Δ (Theorem 1), a negative answer to Problem 10 (Theorem 1), and an affirmative answer to Conjecture 2.9 (Theorem 2).

THEOREM 1. *There exists a weight* w *in* Δ *and a function* $f \in \mathbf{C}(\Delta)$ *such that* $w \geqslant v$ *in* Δ *and*

$$\limsup_{n\to\infty} \|f - L_n(w, f)\|_{v,p} = \infty, \qquad \forall p > 2. \tag{1}$$

In particular,

$$\limsup_{n\to\infty} \|f - L_n(w, f)\|_{w,p} = \infty, \qquad \forall p > 2, \tag{2}$$

and

$$\limsup_{n\to\infty} \|f - L_n(w, f)\|_{p} = \infty, \qquad \forall p > 4. \tag{3}$$

THEOREM 2. *There exists a weight* w *in* Δ *such that* $w \geqslant v$ *in* Δ *and*

$$\limsup_{n\to\infty} \|p_n(w)\|_{v,p} = \infty \tag{4}$$

for every $p > 2$.

Note that Theorem 2 delivers more than what Conjecture 2.9 says since $\{p_n(w)\}_{n=0}^{\infty}$ turns out to be unbounded in L_v^p not just in L_w^p.

Theorem 3 gives a useful relationship between mean boundedness of orthogonal polynomials and mean convergence of the corresponding Lagrange interpolation process.

THEOREM 3. *Let $0<p<\infty$, and let w and u be weights in Δ. Suppose that* $\lim_{n\to\infty} \|f-L_n(w,f)\|_{u,p}=0$ *holds for all $f \in \mathbf{C}(\Delta)$. Then*

$$\sup_{n\in\mathbb{N}} n^{-2/p} \|p_n(w)\|_\infty < \infty$$

if $u \geqslant 1$ in Δ, and

$$\sup_{n\in\mathbb{N}} n^{-1/p} \|p_n(w)\|_\infty < \infty$$

if $u \geqslant v$ in Δ, respectively.

The proofs of these theorems are based on the following lemmas.

LEMMA 1 [8, Theorem 12, p. 324]. *Let $0<p<\infty$, and let w and u be weights in Δ. Then*

$$\sup_{n\in\mathbb{N}} \frac{\|p_n(w)\|_{u,p}}{\|L_n(w)\|_{u,p}} < \infty.$$

The following Nikol'skiĭ-type inequalities are well known. They are a special case of [3, Theorem 6.3.13, p. 113] where the case of more general Jacobi weights is dealt with.

LEMMA 2. *Let $0<p<\infty$. Given $n \in \mathbb{N}$, let Q be a polynomial of degree n. Then there is a constant $c>0$ depending on p only such that*

$$\|Q\|_\infty \leqslant cn^{2/p} \|Q\|_{1,p} \qquad \text{and} \qquad \|Q\|_\infty \leqslant cn^{1/p} \|Q\|_{v,p}. \tag{5}$$

The following two extensions of the *uniform boundedness principle* proved to be quite useful when constructing universal examples for the divergence of some approximation processes.

LEMMA 3 [3, Theorem 10.19, p. 182]. *Let $0<p<\infty$, and let $\mathbb{N}_0 \subseteq \mathbb{N}$ with* $\mathrm{card}(\mathbb{N}_0)=\infty$. *Let w and u be weights in Δ. If* $\lim_{n\in\mathbb{N}_0} \|f-L_n(w,f)\|_{u,p}=0$ *for all $f\in\mathbf{C}(\Delta)$, then* $\sup_{n\in\mathbb{N}_0} \|L_n(w)\|_{u,p}<\infty$.

LEMMA 4 [4, Lemma, p. 191]. *Let $0<s_0<\infty$. Let D be a Banach space with norm $\|\cdot\|$ and let $\{B_s\}_{s_0<s\leqslant\infty}$ be a collection of Banach spaces B_s with*

norms $\|\cdot\|_s$ *such that* $B_s \subseteq B_t$ *for* $s > t$, *and* $\|b\|_t \leqslant \|b\|_s$ *if* $t < s$ *and* $b \in B_s$. *Let* $\{L_m: D \to B_\infty\}_{m \in \mathbb{N}}$ *be a sequence of bounded linear operators such that* $\lim_{m \to \infty} \sup_{\|f\| \leqslant 1} \|L_m(f)\|_s = \infty$ *for every* $s_0 < s \leqslant \infty$. *Then there exists* $f \in D$ *such that* $\limsup_{m \to \infty} \|L_m(f)\|_s = \infty$ *for every* $s_0 < s \leqslant \infty$.

The following result was formulated by Rakhmanov in [7, p. 263] where he points out that it can be proved in the same way that [7, Theorem 1′, p. 261] is deduced from [7, Theorem 2′, p. 258].[2]

LEMMA 5. *Let* τ *be one of the endpoints of* Δ, *and let* $\{\delta_n\}_{n \in \mathbb{N}}$ *be a sequence of positive numbers such that* $\lim_{n \to \infty} \delta_n = 0$. *Then there exists a weight* w *in* Δ *and* $\mathbb{N}_1 \subseteq \mathbb{N}$ *with* $\operatorname{card}(\mathbb{N}_1) = \infty$ *such that* $w \geqslant v$ *in* Δ *and* $|p_n(w, \tau)| \geqslant \delta_n n^{1/2} (\log n)^{-3/2}$ *for* $n \in \mathbb{N}_1$.

Proof of Theorem 3. By Lemma 3, we have $\sup_{n \in \mathbb{N}} \|L_n(w)\|_{u,p} < \infty$. Hence, by Lemmas 1 and 2 the theorem follows. ▮

Proof of Theorem 2. Let w be chosen as in Lemma 5 with $\delta_n = (1 + \log n)^{-1}$. By (5), $\|p_n(w)\|_\infty \leqslant c n^{1/p} \|p_n(w)\|_{v,p}$ for $n \in \mathbb{N}$. Hence, by Lemma 3, formula (4) holds for every $p > 2$. ▮

Proof of Theorem 1. By Theorem 2, there exists w such that $w \geqslant v$ in Δ and (4) holds for every $p > 2$. By Lemma 1, $\limsup_{n \to \infty} \|L_n(w)\|_{v,p} = \infty$ for every $p > 2$. Hence, by Lemma 4, there is a function $f \in \mathbf{C}(\Delta)$ such that (1) holds for every $p > 2$. Formula (2) follows from (1). To prove (3), given $p > 4$, choose $\varepsilon > 0$ such that $p(1 - \varepsilon) > 4$. By Hölder's inequality

$$\int_\Delta |f - L_n(w, f)|^{p(1-\varepsilon)/2} v \leqslant \left\{ \int_\Delta |f - L_n(w, f)|^p \right\}^{(1-\varepsilon)/2} \left\{ \int_\Delta v^{2/(1+\varepsilon)} \right\}^{(1+\varepsilon)/2}.$$

Note that $2/(1+\varepsilon) < 2$ so that $v^{2/(1+\varepsilon)} \in L^1(\Delta)$. Since $p(1-\varepsilon)/2 > 2$, (3) follows from (1). ▮

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[2] More specifically, he means to point out since what is actually written is a little different. In addition, although it is not stated explicitly in [7, Theorem 2′, p. 258], Rakhmanov's weight is symmetric with respect to $\mathbb{R}$ so that the transition from the circle to Δ is straightforward (cf. [7, p. 261, lines 2 and 3 from the top]).

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